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The NBI matrix model of IIB Superstrings

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Abstract: We investigate the NBI matrix model with the potential $X\Lambda + X^{-1} + (2\eta + 1)\log X$ recently proposed to describe IIB superstrings. With the proper normalization, using Virasoro constraints, we prove the equivalence of this model and the Kontsevich matrix model for $\eta \neq 0$ and find the explicit transformation between the two models

Keywords: 1/N Expansion, Matrix Models.

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Contents

1. Introduction

The investigation of matrix models with an external field and logarithmic potentials was initiated in [1, 2]. These matrix models are related to the so-called NBI matrix model, which appeared recently in the context of the IIB superstring matrix model [3, 4, 5, 6]. In [5, 6], the following (M)atrix model action was proposed:

$$S_{\mathrm{NBI}} = -\frac{\alpha}{4}\,\mathrm{tr}\,Y^{-1}[A_\mu, A_\nu]^2 + \beta\,\mathrm{tr}\,Y + 2\eta\,\mathrm{tr}\,\log Y - \frac{1}{2}\,\mathrm{tr}\,\overline{\Psi}\Gamma^\mu[A_\mu,\Psi]\,. \tag{1.1}$$

It possesses $\mathcal{N}=2$ supersymmetry in the large N limit [5]. The matrix Y plays here the role of the world-sheet metric to be integrated out in order to obtain the effective action. Then, the (nonlocal) logarithmic term is a curvature term of the world-sheet metric. As was shown in [2], the relevant choice of the constant η in front of this term that leads to a non-Abelian Born–Infeld action for the string coordinates is $\eta = -\frac{1}{4N}$, N being the matrix size. However, theories for other values of η were also considered [7], and the answer for the effective action in the leading order in N was obtained in [2] for general η.

In the present paper, we derive the constraint equations for the NBI matrix model, and show their coincidence with the constraint equations for the Kontsevich matrix model, thereby proving the equivalence of the two models for nonzero η.

2. The matrix model in the large N limit

We start with the following matrix integral:

$$Z = \int dX \; \mathrm{e}^{-N \,\mathrm{tr}\left[X\Lambda + X^{-1} + (2\eta+1)\log X\right]} . \tag{2.1}$$

As shown in [2], it is related to the bosonic part of (1.1) by the following change in integration variables: $Y = \frac{N}{\beta} X^{-1}$, and by

$$\Lambda = -\frac{\alpha\beta}{4N^2}\,[A_\mu, A_\nu]^2 . \tag{2.2}$$

The matrix integral (2.1) belongs to a class of generalized Kontsevich models (GKM) [8]. Such models with negative powers of the matrix X have been previously discussed in the context of $c = 1$ bosonic string theory [9]. In [7], the τ-function approach to such models was developed. There, the parameter η plays the role of the zeroth time in the corresponding integrable hierarchy. Moreover, at the conformal point $\eta = 0$, this model was shown [7] to have the same Schwinger–Dyson equations as the $U(N)$ model solved in [10, 11].

For the models of this type, the large N solutions are known explicitly only in some special cases. The models with cubic potential for X [12] and the combination of the logarithmic and quadratic potentials [1] were solved by a method based on Schwinger–Dyson equations, developed first for the unitary matrix models with external field [10, 11]. The same technique, being applied to the integral (2.1), also allows one to find its large N asymptotic expansions in the closed form for arbitrary η [2].

The Schwinger–Dyson equations for (2.1) follow from the identity

$$\frac{1}{N^3}\frac{\partial}{\partial \Lambda_{jk}}\frac{\partial}{\partial \Lambda_{li}} \int dX \; \frac{\partial}{\partial X_{ij}} \; \mathrm{e}^{-N \,\mathrm{tr}\left[X\Lambda + X^{-1} + (2\eta+1)\log X\right]} = 0 . \tag{2.3}$$

Written in terms of the eigenvalues, these N equations read (no summation over i is implied)

$$\left[-\frac{1}{N^2}\,\lambda_i \frac{\partial^2}{\partial\lambda_i^2} - \frac{1}{N^2}\sum_{j\neq i}\lambda_j \frac{1}{\lambda_j - \lambda_i}\left(\frac{\partial}{\partial\lambda_j} - \frac{\partial}{\partial\lambda_i}\right) + \frac{1}{N}\,(2\eta - 1)\frac{\partial}{\partial\lambda_i} + 1\right] Z(\lambda) = 0 . \tag{2.4}$$

For $\eta = 0$, these formulas coincide with the corresponding formulas for the $U(N)$ model [10, 11].

It is convenient to set

$$W(\lambda_i) = \frac{1}{N}\frac{\partial}{\partial\lambda_i}\log Z . \tag{2.5}$$

This quantity plays an important role in evaluating the large N limit.

We also introduce the eigenvalue density of the matrix Λ:

$$\rho(x) = \frac{1}{N}\sum_i \delta(x - \lambda_i) . \tag{2.6}$$

The density obeys the normalization condition

$$\int dx\, \rho(x) = 1\,, \tag{2.7}$$

and in the large N limit it becomes a smooth function.

Simple power counting shows that the derivative of $W(\lambda_i)$ in the first term on the left-hand side of Eq. (2.4) is suppressed by the factor $1/N$ and can be omitted at $N = \infty$. The remaining terms are as follows:

$$-xW^2(x) - \int dy\, \rho(y)\, y\, \frac{W(y) - W(x)}{y - x} + (2\eta - 1)W(x) + 1 = 0\,, \tag{2.8}$$

where λ_i is replaced by x. Equation (2.8) can be simplified by the substitution

$$\tilde{W}(x) = xW(x) - \eta\,. \tag{2.9}$$

After some transformations, using the normalization condition (2.7), we obtain

$$\tilde{W}^2(x) + x \int dy\, \rho(y)\, \frac{\tilde{W}(y) - \tilde{W}(x)}{y - x} = x + \eta^2\,. \tag{2.10}$$

The nonlinear integral equation (2.10) can be solved with the help of the ansatz

$$\tilde{W}(x) = f(x) + \frac{x}{2} \int dy\, \frac{\rho(y)}{f(y)}\, \frac{f(y) - f(x)}{y - x}\,, \tag{2.11}$$

where $f(x)$ is an unknown function to be determined by substituting (2.11) into Eq. (2.10). The asymptotic behaviors of $\tilde{W}(x)$ and $f(x)$ as $x \to \infty$ follow from eq. (2.10): $\tilde{W}(x) \sim \sqrt{x} - 1/2$, and the analytic solution with minimal set of singularities is

$$f(x) = \sqrt{ax + b}\,. \tag{2.12}$$

The parameters a and b are unambiguously determined from Eq. (2.10). We find that $b = \eta^2$ and a is implicitly defined by

$$1 + \frac{1}{2} \int dy\, \frac{\rho(y)}{f(y)} = \frac{1}{\sqrt{a}}\,, \tag{2.13}$$

or, in terms of the eigenvalues,

$$1 + \frac{1}{2N} \sum_j \frac{1}{\sqrt{a\lambda_j + \eta^2}} = \frac{1}{\sqrt{a}}\,. \tag{2.14}$$

Then, the answer for the integral in the large N limit reads [2]

$$\log Z = N^2 \left[\left(\eta^2 + \frac{1}{4} \right) \log a + \frac{4\eta^2}{\sqrt{a}} - \frac{\eta^2}{a} \right] + \\ +N \sum_i \left[\frac{2}{\sqrt{a}} \sqrt{a\lambda_i + \eta^2} + \eta \log \left(\lambda_i \frac{\sqrt{a\lambda_i + \eta^2} - \eta}{\sqrt{a\lambda_i + \eta^2} + \eta} \right) \right] - \\ +\frac{1}{2} \sum_{ij} \log \left(\sqrt{a\lambda_i + \eta^2} + \sqrt{a\lambda_j + \eta^2} \right). \tag{2.15}$$

One can verify directly that $\frac{\partial}{\partial a} \log Z = 0$ and $\frac{1}{N} \frac{\partial}{\partial \lambda_i} \log Z = W(\lambda_i)$, as far as Eq. (2.14) holds.

3. The Kontsevich phase

We are interested in the asymptotic expansion of the model (2.1) for large Λ. Then, the expansion parameters are traces of negative powers of the external matrix Λ. Conventionally, this regime is called the *Kontsevich phase* of the solution.

Here an important note is in order. In [2], we did not discuss which branch of the root—positive or negative—should be chosen in (2.14), since both choices led to the same answer for the integral in the large N limit (2.15). However, in what follows, we must fix this sign.

The Kontsevich phase is the strong coupling regime where the expansion in negative powers of λ_i is to be performed. Then we see that the dependence is only on $\lambda_i^{-n-1/2}$, $n = 0, 1, \ldots$

As the first step, we perform the phase analysis for the toy case where all λ_i's coincide, so (2.14) becomes

$$\frac{a}{(1-\sqrt{a})^2} = 4(a\lambda + \eta^2), \quad a > 0\,,\ \lambda > 0\,. \tag{3.1}$$

Then, obviously, the sign of the square term in (2.14) is negative for $a > 1$ and positive for $0 < a < 1$.

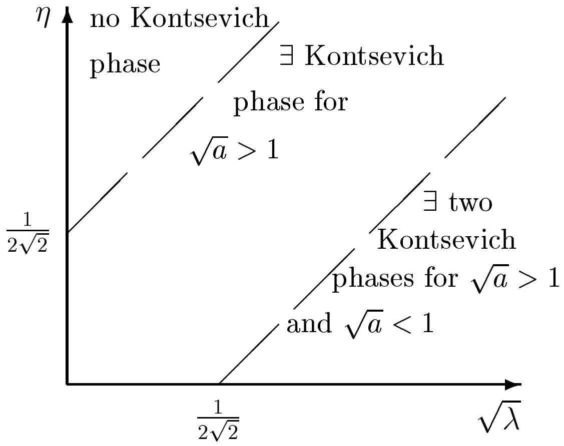
Algebraically, there always (except if $\eta = 0$) exist two solutions to (3.1): one with $0 < a < 1$ and another with $a > 1$. For the Kontsevich phase to be possible, we demand that the expansion in terms of the so-called *times*

$$\tau_k = \frac{1}{2k-1} \operatorname{tr} \Lambda^{-k+1/2}, \quad k = 1, 2, \ldots \tag{3.2}$$

should make sense. So, we assume that

$$\left| \frac{\eta^2}{a\lambda} \right| < 1\,. \tag{3.3}$$

Then, from (3.1) and (3.3), we obtain the following phase diagram:



4. Constraint equations in the Kontsevich phase

Now we write (2.4) in terms of the relevant times (3.2). Here, to obtain rigorous results, the normalizing factor is necessary. From the theory of the generalized Kontsevich model [8], the proper expression, which has no explicit dependence on the matrix size N, reads

$$\mathcal{Z}(\{\tau_n\}) = \frac{\int DX\, \mathrm{e}^{\,\Lambda X + V(X)}}{\mathrm{e}^{\,\mathrm{tr}\,\Lambda X_0 + V(X_0)} \det^{-1/2}\left(\frac{\delta}{\delta X} \otimes \frac{\delta}{\delta X} V(X_0)\right)} \,. \tag{4.1}$$

Here X_0 is the stationary point, $\Lambda + V'(X_0) = 0$, and the determinant in the normalizing factor comes from the quasi-classical integration.

Let us choose the new variables

$$\lambda_i = z_i^2 - (\eta + 1/2)^2 \,. \tag{4.2}$$

Then the normalizing factor reads

$$\exp\left\{ -N \sum_i \left[2z_i - 2\eta \log(z_i + \eta + 1/2)\right] - \frac{1}{2} \sum_{i,j} \log(z_i + z_j) \right\} . \tag{4.3}$$

Note that if we perform the standard Itzykson–Zuber integration, expression (4.1) for integral (2.1) becomes

$$\mathcal{Z}(\{\tau_n\}) = \frac{\det\limits_{\substack{1 \le i \le N \\ 0 \le l \le N-1}} \left|\left| \xi_i^{2\eta N - l} K_{-2\eta N - l}(N\xi_i) \right|\right|}{\prod_i \mathrm{e}^{\,-2Nz_i} (z_i + \eta + 1/2)^{2\eta N} (2z_i)^{-1/2} \Delta(z)} \,, \qquad \xi_i \equiv \sqrt{\lambda_i}, \tag{4.4}$$

where $K_\nu(x)$ are Macdonald functions, $K_\nu(x) = \int_{-\infty}^{\infty} ds\, \mathrm{e}^{\,-2x\cosh s + \nu s}$, and (4.4) does not resemble too much the corresponding expression for the Kontsevich matrix model where Airy functions stand instead of $K_\nu(x)$. To find the large N asymptotic behavior of (4.4), we use the constraint equation method.

In terms of z-variables, (2.4) becomes

$$\left[-\frac{1}{N^2}\left(z_i^2-(\eta+1/2)^2\right)\frac{1}{2z_i}\frac{\partial}{\partial z_i}\left(\frac{1}{2z_i}\frac{\partial}{\partial z_i}\right)-\frac{1}{N^2}\sum_{j\neq i}\frac{z_j^2-(\eta+1/2)^2}{z_j^2-z_i^2}\times\right.$$
$$\left.\times\left(\frac{1}{2z_j}\frac{\partial}{\partial z_j}-\frac{1}{2z_i}\frac{\partial}{\partial z_i}\right)+\frac{2\eta-1}{N}\frac{1}{2z_i}\frac{\partial}{\partial z_i}+1\right]Z(z_i)=0\,. \quad (4.5)$$

When pushing the normalizing factor (4.3) through derivatives w.r.t. z_j-variables, we replace $\left(\partial_i\equiv\frac{\partial}{\partial z_i}\right)$

$$\partial_i\to\partial_i-2N+2\eta N\frac{1}{z_i+\eta+1/2}-\sum_j\frac{1}{z_i+z_j}\,. \quad (4.6)$$

Let us introduce the new *times*

$$t_{n+1}=\frac{1}{2n-1}\sum_i\frac{1}{z_i^{2n-1}}+\delta_{n,0}\frac{N}{\eta+1/2}\,,\qquad n=0,1,\ldots, \quad (4.7)$$

which differs slightly from the conventional ones defines above by (3.2). As is easily checked, they are related by a lower triangular transformation, so they are equivalent from the view point of phase transitions and critical behavior.

Then, the constraint equations for $\mathcal{Z}(\{t_n\})$ are obtained after some tedious algebra which we omit here. Collecting all coefficients to the term $\frac{1}{z_i^{2s+4}}\equiv\frac{\partial}{\partial z_i}t_{s+2}$, we obtain

$$\widetilde{L}_s\mathcal{Z}(\{t_n\})=0\,,\qquad s\geq-1\,, \quad (4.8)$$

where

$$\widetilde{L}_s=\delta_{s,-1}\left[-\frac{1}{16N^2}+t_1^2\frac{(\eta+1/2)^2}{4N^2}\right]+\delta_{s,0}\frac{(\eta+1/2)^2}{16N^2}-$$
$$-\frac{1}{2N^2}\sum_{\substack{p=0\\p+s\geq0}}^{\infty}\left[-(\eta+1/2)^2(2p+1)t_{p+1}+(2p-1)(1-\delta_{p,0})t_p\right]\frac{\partial}{\partial t_{p+s+1}}-$$
$$-\frac{1}{N}\left(\frac{\partial}{\partial t_{s+2}}-\frac{\eta+1/2}{2}(1-\delta_{s,-1})\frac{\partial}{\partial t_{s+1}}\right)-$$
$$-\frac{1}{4N^2}\sum_{k=1}^{s+1}\frac{\partial}{\partial t_k}\frac{\partial}{\partial t_{s+2-k}}+\frac{(\eta+1/2)^2}{4N^2}\sum_{k=1}^{s}\frac{\partial}{\partial t_k}\frac{\partial}{\partial t_{s+1-k}}\,. \quad (4.9)$$

In the convenient conformal field theory notation, (4.9) becomes

$$\widetilde{L}_s = V_{s+1} - (\eta + 1/2)^2 V_s + \frac{4\eta N}{\eta + 1/2}\alpha_{2s+3}\,. \tag{4.10}$$

where

$$\alpha_{2n+1}\Big|_{n\geq 0} \equiv \frac{\partial}{\partial t_{n+1}}\,, \tag{4.11}$$

$$\alpha_{-2n-1}\Big|_{n\geq 0} \equiv (2n+1)t_{n+1}\,, \tag{4.12}$$

$$[\alpha_{-2k-1}, \alpha_{2q+1}] = -(2k+1)\delta_{k,q}\,, \tag{4.13}$$

and the operators V_s in the free-field representation are quadratic in α_i,

$$V_q \equiv \sum_{a,b} \delta_{q,a+b+1}{:}\alpha_{2a+1}\alpha_{2b+1}{:} + 1/4\delta_{s,0}\,, \tag{4.14}$$

where the normal ordering : : means that all α_a with positive indices are on the right of all α_b with negative indices.

For $s, t \geq -1$, $\widetilde{L}_s$ satisfy the algebra

$$[\widetilde{L}_s, \widetilde{L}_t] = 4(s-t)\left(\widetilde{L}_{s+t+1} - (\eta + 1/2)^2\widetilde{L}_{s+t}\right), \tag{4.15}$$

from which the Virasoro algebra can be obtained by the lower-triangle replacement

$$L_s \equiv \widetilde{L}_s + \sum_{k=1}^{\infty}(\eta + 1/2)^{-2k}\widetilde{L}_{s+k}\,. \tag{4.16}$$

Performing replacement (4.16) and rescaling V_s, we obtain the Virasoro algebra in terms of the generators

$$L_s = -(\eta + 1/2)^2 V_s + 4\eta N\sum_{k=2}^{\infty}\frac{1}{(\eta + 1/2)^{2k-3}}\partial_{s+k}\,. \tag{4.17}$$

Amazingly, if we manage to remove the derivative terms in (4.17), then the constraints we obtain will be just constraints of the Kontsevich matrix model [13, 14]. To remove these terms, we shift all of the higher times, leaving the times t_1 and t_2 unshifted,

$$\tilde{t}_k \equiv t_k - \frac{4\eta N}{(\eta + 1/2)^{2k+1}}\frac{1}{2(2k+1)}\,, \qquad k \geq 3\,, \qquad \tilde{t}_{1,2} = t_{1,2}\,. \tag{4.18}$$

Explicitly, in terms of the new times,

$$L_{-1} = \tilde{t}_1^2 + 2\sum_{m=1}^{\infty}(2m+1)\tilde{t}_{m+1}\frac{\partial}{\partial\tilde{t}_m} - \frac{4\eta N}{(\eta + 1/2)^3}\partial_1\,, \tag{4.19}$$

$$L_s = \sum_{m=1}^{s}\frac{\partial}{\partial\tilde{t}_m}\frac{\partial}{\partial\tilde{t}_{s-m}} + 2\sum_{m=1}^{\infty}(2m-1)\tilde{t}_m\frac{\partial}{\partial\tilde{t}_{m+s}} + \frac{1}{4}\delta_{s,0} - \frac{4\eta N}{(\eta + 1/2)^3}\frac{\partial}{\partial\tilde{t}_{s+2}}\,. \tag{4.20}$$

This is nothing but the Virasoro algebra that appears in the Kontsevich matrix model. Therefore, we have proven the equivalence between the model (2.1) and the Kontsevich matrix model [15].

5. A large N limits comparison

Let us explicitly compare the model (2.1) after the time changing (4.18) and the Kontsevich model with the partition function

$$Z(M)=\frac{\int dX \exp - \operatorname{tr}\gamma\{MX^2/2-iX^3/6\}}{\int dX \exp - \operatorname{tr}\gamma MX^2/2}\,,\qquad M=\operatorname{diag}\{m_1,\ldots,m_N\}\,. \tag{5.1}$$

Let us consider the constraint equation (2.14). We have

$$\frac{1}{2n-1}\operatorname{tr}M^{-2n+1}=\tilde{t}_n=\frac{1}{2n-1}\operatorname{tr}Z^{-2n+1}+\begin{cases}\frac{N}{\eta+1/2}\,, & n=1\,,\\ 0\,, & n=2\,,\\ -\frac{1}{2(2n-1)}\frac{4\eta N}{(\eta+1/2)^{2n-1}}\,, & n\geq 3\,,\end{cases} \tag{5.2}$$

i.e.,

$$\operatorname{tr}M^{-2n+1}=\operatorname{tr}Z^{-2n+1}-\frac{1}{2}\frac{4\eta N}{(\eta+1/2)^{2n-1}}+\frac{1}{2}\frac{4\eta N}{(\eta+1/2)^{3}}\delta_{n,2}+2N\delta_{n,1}\,. \tag{5.3}$$

Then we obtain, in terms of these shifted variables, ($\pm$ depends on the branch of the square root),

$$\begin{aligned}\frac{1}{\sqrt{a}}&=1\pm\frac{1}{2N}\sum_j\frac{1}{\sqrt{az_j^2-a(\eta+1/2)^2+\eta^2}}\\&=1\pm\frac{1}{2N\sqrt{a}}\sum_j\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{z_j^{2n+1}}((\eta+1/2)^2-\eta^2/a)^n\\&=1\pm\frac{1}{2N\sqrt{a}}\Bigg[\sum_j\frac{1}{\sqrt{m_j^2+\eta^2/a-(\eta+1/2)^2}}+\\&\qquad\qquad +2N\sqrt{a}-2N-\frac{\eta N}{\eta+1/2}+\frac{\eta^3 N}{(\eta+1/2)^3 a}\Bigg].\end{aligned} \tag{5.4}$$

Assuming the minus sign and denoting

$$(\eta+1/2)^2-\eta^2/a\equiv 2s\,,\qquad -\frac{2\eta N}{(\eta+1/2)^3}\equiv\gamma\,, \tag{5.5}$$

we obtain

$$\sum_{i=1}^{N}\frac{1}{\sqrt{m_i^2-2s}}=-\gamma s\,, \tag{5.6}$$

i.e., the constraint equation of the Kontsevich model itself [13, 14].

Also, let us recall the answer in the large N limit for the Kontsevich model [13]. If we reconstruct the dependence on the coupling constant γ, then it reads (in the original notation)

$$F_0 = \gamma^2 \frac{s^3}{6} + \frac{\gamma}{3} \sum_{i=1}^{N} \left\{ m_i^3 - (m_i^2 - 2s)^{3/2} - 3s\sqrt{m_i^2 - 2s} \right\} - \\ -\frac{1}{2} \sum_{i,j=1}^{N} \log \frac{\sqrt{m_i^2 - 2s} + \sqrt{m_j^2 - 2s}}{m_i + m_j}, \qquad (5.7)$$

where s is determined from the constraint equation (5.6).

Let us compare the answer (5.7) with formula (2.15) while accounting for the normalizing condition (4.3). Then, in variables z_i, assuming the minus sign in the constraint (2.14), we have

$$\widetilde{F}_0 = N^2\eta^2 \log a + \frac{4N^2\eta^2}{\sqrt{a}} - \frac{N^2\eta^2}{a} + N \sum_i 2\left(z_i - \sqrt{z_i^2 - 2s}\right) + \\ + N\eta \sum_i \log \left(\frac{z_i^2 - (\eta + 1/2)^2}{(z_i + \eta + 1/2)^2} \right) \left(\frac{-\sqrt{z_i^2 - 2s} - \eta/\sqrt{a}}{-\sqrt{z_i^2 - 2s} + \eta/\sqrt{a}} \right) + \\ -\frac{1}{2} \sum_{i,j} \log \left(\frac{\sqrt{z_i^2 - 2s} + \sqrt{z_j^2 - 2s}}{z_i + z_j} \right). \qquad (5.8)$$

After a tedious algebra, taking into account (5.3), we obtain

$$\widetilde{F}_0 = F_0 + 2N^2\eta^2 \log \frac{\eta}{\eta + 1/2} + N^2 \left[\frac{7}{3}\eta^2 + \eta - \frac{1}{4} \right], \qquad (5.9)$$

where F_0 is given by (5.7) with the substitution (5.5). The difference between $\widetilde{F}_0$ and F_0 is just irrelevant constant terms. This again proves that the two matrix models under consideration coincide.

6. Higher genus expressions

In this section, we set the Kontsevich coupling constant $\gamma = 1$. The higher genus contributions in the Kontsevich model are expressed in terms of the so-called moments I_k,

$$I_0 = -\frac{1}{N} \sum_{i=1}^{N} \frac{1}{\sqrt{m_i^2 - 2s}}, \qquad I_k = \frac{\partial^k}{\partial s^k} I_0, \quad k > 0. \qquad (6.1)$$

Then, for $g > 1$, we have [14]

$$F_g = \sum_{\sum_{k=2}(k-1)l_k = 3g-3} \langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{3g-2}^{l_{3g-2}} \rangle_g \frac{1}{(1 - I_1)^{2(g-1)+\sum l_p}} \frac{I_2^{l_2}}{l_2!} \frac{I_3^{l_3}}{l_3!} \cdots \frac{I_{3g-2}^{l_{3g-2}}}{l_{3g-2}!}, \qquad (6.2)$$

i.e., F_g is a finite sum of monomials in $I_k/(1-I_1)^{(2k+1)/3}$ with coefficients being the intersection indices on the moduli space [15]. For $g=1$, $F_1 = \frac{1}{24}\log\frac{1}{1-I_1}$.

Now we rewrite the expression (6.2) via the moments of the model (2.1). Let us introduce the new moments J_k

$$J_k = -(2k-1)!!\frac{1}{N}\sum_{j=1}^{N}\frac{1}{(a\lambda_j+\eta^2)^{k+1/2}}, \qquad k=0,1,2,\ldots \tag{6.3}$$

We are interested only in transformation law for I_k with $k>0$ since the only dependence on the moment I_0 is via the constraint equation (5.6). Then, Eq. (5.3) implies

$$\begin{aligned}(I_1-1) &\to a^{3/2}(J_1+2/\eta^2)\,,\\ I_k &\to a^{k+1/2}(J_k+2/\eta^{2k})\,,\end{aligned}$$

i.e., the expression (6.2) becomes

$$F_g^{NBI} = \sum_{\sum_{k=2}(k-1)l_k=3g-3}\langle\tau_2^{l_2}\tau_3^{l_3}\cdots\tau_{3g-2}^{l_{3g-2}}\rangle_g\frac{1}{(J_1+\frac{2}{\eta^2})^{2(g-1)+\sum l_p}}\prod_{k=2}^{3g-2}\frac{(J_k+\frac{2}{\eta^{2k}})^{l_k}}{l_k!}, \quad g>1 \tag{6.4}$$

and

$$F_1^{NBI} = \frac{1}{24}\log\left[a^{3/2}(J_1+2/\eta^2)\right]. \tag{6.5}$$

Therefore, expression (2.15) for genus zero, taking into account the normalizing factor (4.3) and the expressions (6.5), (6.4) completely determine the partition function of the model (2.1) for all genera. These expansions are, however, ill-defined for $\eta\to 0$, which corresponds to the $U(N)$ model, and for $\eta\sim 1/N$, which corresponds to the model defined in [5].

Remark 1 The last observation above is related to the initial constraints (4.8). Note that we can consider a "minimal reduction," where all times t_k but the time t_1 are equal to zero. Then this partition function is entirely determined from the action of $\widetilde{L}_{-1}$ on $\mathcal{Z}$,

$$\left(\frac{4\eta}{\eta+1/2}+2t_1\right)\frac{\partial\mathcal{F}}{\partial t_1}\bigg|_{t_2=t_3=\ldots=0} - N^2(\eta+1/2)^2t_1^2+1/4=0\,,$$

where

$$t_1 \equiv \frac{1}{N}\sum_{i=1}^{N}z_i^{-1}+(\eta+1/2)^{-1}|_{t_2=t_3=\ldots=0} = \frac{1}{N}\sum_{i=1}^{N}\lambda_i^{-1/2}+(\eta+1/2)^{-1}\,,$$

i.e.,

$$\frac{\partial\mathcal{F}}{\partial t_1}\bigg|_{t_2=t_3=\ldots=0} = \frac{\eta+1/2}{4}\left[N^2((\eta+1/2)t_1-2\eta)+\frac{(2\eta N)^2-1/4}{(\eta+1/2)t_1+2\eta}\right]. \tag{6.6}$$

Remark that $\mathcal{F}$ becomes a polynomial in t_1 for

$$\eta = \pm\frac{1}{4N}\,,$$

and this is exactly the point corresponding to the IIB superstring model [5].

On the other hand, from expressions (6.4), (6.5) it is clear that in this case all $J_k = 0$ for $k > 0$, so there is no t_1-dependent contributions from $g > 1$ and only $\frac{1}{16}\log a$ comes from the genus one term. One can compare with expression for F_1 coming from (6.6) and, taking into account the constraint equation (2.14), one finds an exact coincidence of the two expressions, i.e.,

$$F_1|_{t_2=t_3=\ldots=0} = -1/8\log(2+1/N\,\mathrm{tr}\,\lambda^{-1/2}) = \frac{1}{16}\log a\,.$$

Remark 2 The string susceptibility in the large N limit can be obtained by differentiating twice w.r.t. the string coupling constant η. One should check that after the first differentiation of (2.15), the stationary condition still holds, so the total derivative coincides with the partial one and, as a result, we have

$$\frac{\mathrm{d}^2\log Z}{\mathrm{d}\eta^2} = \frac{\partial^2\log Z}{\partial\eta^2} = 2N^2(\log a + 3)\,. \tag{6.7}$$

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